MATHEMATICS

TWO-POINT HOMOGENEOUS MANIFOLDS

BY

H. O. SINGH VARMA [1])

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of April 24, 1965)

Introduction

In [10] H. C. WANG has determined the compact two-point homogeneous Riemannian manifolds. (See the definition below.) His results imply that these spaces are globally symmetric, and hence of rank one [5]. It is the purpose of this paper to provide an alternate method of establishing the fact that two-point homogeneous manifolds are symmetric. The classification of all groups acting transitively on spheres is not used, in contrast to [10], but as in [10], the proof depends on the results of [2]. Also a theorem of R. BOTT [3], plays an important role. It must be pointed out that we have profited much from WANG's work [10], to which the present paper is but a small appendix.

1. *Some properties of two-point homogeneous manifolds*

Definition: Let M be a connected, compact Riemannian manifold with distance function d. M is said to be a *two-point homogeneous* manifold if for any two pairs (p, q) and (p', q') of points in M, satisfying the condition $d(p, q)=d(p', q')$, there is an isometry φ of M, such that $\varphi(p)=p'$ and $\varphi(q)=q'$.

M is said to be a *(*)-manifold,* if there is a positive real number r, such that for any two pairs (p, q) and (p', q') of points in M that satisfy the conditions:

$$d(p, q)\leqslant r \quad ; \quad d(p', q')\leqslant r \quad ; \quad d(p, q)=d(p', q')$$

there is an isometry φ of M, such that $\varphi(p)=p'$, $\varphi(q)=q'$.

It is clear that a two-point homogeneous manifold is also a (*)-manifold. Also it is to be noted that a (*)-manifold is certainly homogeneous in the sense that the group $\bar{G}$ of all isometries of M acts transitively on M. It is well-known that the group $\bar{G}$ of all isometries is a compact Lie group, and moreover G, the identity component of $\bar{G}$, also acts transitively on M. (Since M is connected).

---

[1]) The author holds a NATO-fellowship, granted by Z.W.O., the Netherlands organization for the advancement of pure research.

We shall introduce some notation which will be used in the rest of this paper.

(i) M stands for a (connected, compact) (*)-manifold. We put $m = \dim. M$, and suppose $m \geqslant 2$, the cases $m=0$ or 1 being trivial since then M is either a point or a circle.

(ii) $\bar{G}$ will denote the Lie group of isometries of M, and G its identity component.

(iii) We fix a point p in M, and denote by $\bar{H}$ the isotropy–subgroup of $\bar{G}$ at p. Then $M = \bar{G}/\bar{H}$.

(iv) U denotes an ε-ball about p with the following property: for any two points $x \in U$, $y \in U$ there is a geodesic γ of length $d(x, y)$ joining x to y in U, and this geodesic γ is the unique geodesic of length $d(x, y)$ joining x to y in M. [11]

(v) The tangent space to M at p is denoted by M_p.

Now let η be a real number such that $0<\eta<\varepsilon<r$. Since M is a (*)-manifold, $\bar{H}$ acts transitively on the sphere $S(p, \eta) = \{x \in M | d(x, p) = \eta\}$, and because $S(p, \eta)$ is connected ($m \geqslant 2$) the identity component of $\bar{H}$ also acts transitively on $S(p, \eta)$. Therefore we have:

Proposition 1.1. Let M be a (*)-manifold. There exists a positive number η_0 such that for any two pairs of points (p, q) and (p', q') in M, satisfying the conditions

$$d(p, q) \leqslant \eta_0 \ ; \ d(p', q') \leqslant \eta_0 \ ; \ d(p, q) = d(p', q')$$

there exists $\varphi \in G$, such that $\varphi(p) = p'$ and $\varphi(q) = q'$.

This proposition allows us to consider M as a homogeneous space $M = G/H$, where G is a *connected* group of isometries, without losing the (*)-property; we shall do this in the following.

We also note that the linear isotropy group acts transitively on the sphere of radius η $(0<\eta<\varepsilon)$ in M_p. In particular this representation of H, which is faithful, is *irreducible*, and the same is also true true for the linear isotropy representation of H_0, the identity component of H.

Proposition 1.2. The Lie algebra $\mathfrak{H}$ of H is a maximal subalgebra of $\mathfrak{G}$, the Lie algebra of G.

Proof: Let $\mathfrak{H}'$ be a subalgebra of $\mathfrak{G}$ such that $\mathfrak{H} \subset \mathfrak{H}' \subset \mathfrak{G}$, $\mathfrak{H}' \neq \mathfrak{H}$, and let H' be the analytic subgroup of G corresponding to $\mathfrak{H}'$. The orbit $H'. p$ is a submanifold of M, and may be identified with H'/L, where $L \supset H_0$. The tangent space to this orbit at p is a subspace of M_p, invariant under the linear isotropy representation of H_0, so it coincides with M_p. Thus $\dim \mathfrak{H}' - \dim \mathfrak{H} = m$, so $\dim \mathfrak{H}' = \dim \mathfrak{G}$, and $\mathfrak{H}' = \mathfrak{G}$. Q.E.D.

Proposition 1.3. Let M be a (*)-manifold, $M = G/H$ with G connected. Then the geodesics of M are closed, simply closed, and any two geodesics are congruent under an isometry $\varphi \in G$. In particular all geodesics have the same length.

Proof: M is compact, hence complete. We shall assume that geodesics are parametrized by arc length. Now since M is compact, there is a closed non-constant geodesic $\gamma: \mathbf{R} \to M$, $\gamma(s+L)=\gamma(s)$, for $s \in \mathbf{R}$, and L the "length" of γ. See [7]. Let $\gamma(0)=p$. We may suppose that $\gamma(s)=\gamma(0)$ only if s is a multiple of L, and we will say that p is a "simple" point of γ. Now let σ be an arbitrary geodesic starting at p. Choose points q and q' in U such that $q=\gamma(s)$, $q'=\sigma(s)$, $s<\varepsilon$. Then $d(p, q)=d(p, q')=s$. Hence there exists $\varphi \in G$ with $\varphi(p)=p$ and $\varphi(q)=q'$. The isometry φ transforms $\gamma: [0, s] \to M$ into a geodesic of length s joining p to q'. Therefore γ is transformed into σ by φ, i.e. any two geodesics emanating from p are congruent under G. The homogeneity of M then implies that this is true for any two geodesics. To show that γ is simply-closed, let $0<s_0<L$, and put $x=\gamma(s_0)$. Let γ' be the geodesic starting at x, where $\gamma'(s)=\gamma(s+s_0)$. There is an isometry $\varphi \in G$, such that φ takes p into x and γ into γ'. Then since p is a simple point of γ, x is a simple point of γ', i.e. $\gamma(s+s_0)= =\gamma(s_0)$ only if s is a multiple of L. Q.E.D.

Proposition 1.4. A (*)-manifold M is either simply connected or $\pi_1(M) \approx Z_2$.

Proof: If M is not simply connected, any non-trivial element of $\pi_1(M)$ can be represented by a closed geodesic. Since G is connected and any two geodesics are congruent under G, we see that if α and β are non-trivial elements in $\pi_1(M)$, then $\alpha=\beta$. Hence $\pi_1(M)$ consists of two elements. Q.E.D.

In [3] R. Bott has studied Riemannian manifolds V with the following property:

(c): There is a point p in V, such that the geodesics emanating from p are closed, simply closed, and have the same length.

He proves the following

Theorem. (R. Bott). A compact connected Riemannian manifold V, having property (c), is either simply connected or $\pi_1(V) \approx Z_2$. If V is simply connected, the integral cohomology-ring of V is a truncated polynomial ring. If $\pi_1(V) \approx Z_2$, the universal covering $\tilde{V}$ of V is an integral homology-sphere.

We refer to [3] for the proof of this theorem.

For further use we now discuss some properties of homogeneous spaces.

Let L be a compact connected Lie group, K a closed connected subgroup. The homogeneous space $V=L/K$ has non-negative Euler-characteristic: $\chi(V)\geqslant 0$. Moreover $\chi(V)>0$ if and only if K has maximal rank, and in this case $\chi(V)=$ index of $\Phi(K)$ in $\Phi(L)$, if $\Phi(\quad)$ denotes the Weyl group. (Hopf–Samelson [6])

Lemma 1.1. Let L be a compact connected Lie group acting effectively and transitively on the manifold V. If $\chi(V)>0$, then L has trivial center.

Proof: Let K be an isotropy subgroup of L. K has maximal rank, so let T be a maximal torus with $T \subset K \subset L$. The center of L is contained in T, hence the elements of the center of L induce the identity transformation of V. But L acts effectively. Q.E.D.

Lemma 1.2. Let L be a compact connected Lie group acting effectively and transitively on the manifold V. Suppose that the isotropy group at a point $p \in V$ is a maximal connected subgroup of maximal rank. Then L is a centerless simple group.

Proof: From Lemma 1.1. we know that L has trivial center, so the Lie algebra $\mathfrak{G}$ of L is semi-simple, and $L = L_1 \times L_2 \times \ldots \times L_r$ is a direct product of compact connected, non-abelian, simple Lie groups. ($\mathfrak{G}$ is the direct sum of simple ideals.) The subgroup K, the isotropy group at p, is itself a direct product $K = K_1 \times K_2 \times \ldots \times K_r$, where $K_i \subset L_i$ for $i = 1, \ldots, r$. To see this, note that every element of K is on a maximal torus of K, that a maximal torus of K is also a maximal torus of L, since K and L have equal rank, and so is of the form $T_1 \times \ldots \times T_r$, where T_j is a maximal torus of L_j, $(j = 1, \ldots, r)$. This then being the case, at least one of the K_i must be different from the L_i in which it is contained; let us say that $K_1 \neq L_1$. Then $K_1 \times L_2 \times \ldots \times L_r$ is a connected proper subgroup of L containing K, hence it is equal to K. The subgroup $L_2 \times \ldots \times L_r$ of K is normal in L, so its action on $V = L/K$ is the identity. But as L acts effectively, $L_2 \times \ldots \times L_r$ consists only of the identity, and $L = L_1$ is simple. Q.E.D. See Borel [1], Wang [8].

Lemma 1.3. Let L, V be as in Lemma 1.1, $V = L/K$, with K connected. If $\chi(V)$ is a prime number, then K is a maximal connected subgroup of maximal rank. (Borel)

Proof: $\chi(V) =$ index of $\Phi(K)$ in $\Phi(L)$. If N is a proper connected subgroup such that $K \subset N \subset L$, then $\Phi(K) \subset \Phi(N)$, and the groups are different if $K \neq N$. But then $\chi(N/K)$ divides $\chi(V)$, a contradiction. Q.E.D.

Theorem. (A. Borel; H. C. Wang). Let L be a compact connected Lie group, acting effectively and transitively on an even-dimensional sphere S_{2r}. Then L is either isomorphic to $\mathbf{SO(2r+1)}$ or, if $r = 3$, to $\mathbf{G}_2$.

Sketch of proof: $\chi(S_{2r}) = 2$. By Lemma 1.3 and Lemma 1.2 L is simple and the isotropy group (which is connected since S_{2r} is simply connected) has maximal rank and is a maximal connected subgroup. The subgroups of this type for all the compact simple center less Lie groups have been determined in [2]. A. Borel has [1], checked that the only cases where $\Phi(K)$ has index 2 in $\Phi(L)$ are, in the Cartan notation:

(i) $\mathbf{D}_r$ in $\mathbf{B}_r$ $\quad (r = 1, 2, \ldots)$

(ii) $\mathbf{A}_2$ in $\mathbf{G}_2$

The corresponding homogeneous spaces are S_{2r} with the usual action of $\mathbf{SO}(2\mathbf{r}+1)$, and S_6. In this last case $\mathbf{G}_2$ acts on S_6 by considering $\mathbf{G}_2$ as the group of automorphisms of the Cayley algebra, and S_6 as the set of Cayley numbers of norm 1 and real part zero.

Important remark: It is known that in both cases L also acts transitively on the set of two-planes in $\mathbf{R}^{2r+1}$.

2. *The manifolds of odd dimension*

If M has odd dimension and is simply connected, then M is an integral homology-sphere, as follows from Bott's theorem. Also in this case H is connected. If $\pi_1(M) \approx Z_2$, then H is either connected or has two components, and if H has two components we have that $G/H_0 = \tilde{M}$. In all cases G/H_0 is a (*)-manifold and H_0 is a compact connected Lie group acting transitively and effectively on a sphere of even dimension. But then H_0 also acts transitively on the set of two-planes in M_p, so by homogeneity G/H_0 has constant curvature. We now state

Proposition 2.1. If M is an odd-dimensional (*)-manifold, then M is a space of constant curvature.

Proof: $G/H_0 = M$ whenever H is connected. If H has two components G/H_0 is simply connected and has constant curvature, so $\tilde{M}$ is isometric to S_m, and S_m is a two-fold covering of M. But then M also has constant curvature. Q.E.D.

Theorem 2.1. An odd-dimensional two-point homogeneous manifold is a globally symmetric Riemannian manifold.

Proof: If M is simply connected it is isometric to a sphere, hence is globally symmetric. In the other case its two-fold universal covering is isometric to a sphere. But then M is isometric to a real projective space, and thus is globally symmetric. Q.E.D.

3. *The manifolds of even dimension*

If M has even dimension and is simply connected, the integral cohomology-ring of M is a truncated polynomial ring generated by an element of even degree. Hence the Euler characteristic of M is positive.

Lemma 3.1. The Euler characteristic of an even-dimensional (*)-manifold is positive.

Proof: We only have to consider the case where $\pi_1(M) \approx Z_2$. Then $\tilde{M}$, the two-fold universal covering of M, is an even-dimensional homology-sphere, and hence $\chi(\tilde{M}) = 2$. Denoting by $b_k(X)$ the k-dimensional Betti-number of a space X, a well-known theorem of S. Bochner states that $b_k(M) \leqslant b_k(\tilde{M})$ for $0 < k < m$. Since $b_k(\tilde{M}) = 0$ if $k = 1, \ldots, m-1$, we have that $b_k(M) = 0$ if $k = 1, 2, \ldots, m-1$. Now $b_0(M) = 1$, so $\chi(M) \geqslant 1$. Q.E.D.

Remark: The theorem of Bochner used in the last proof can be deduced easily from the following more general theorem due to B. ECKMANN [4].

Let $\tilde{P}$, P be finite polyhedra, $\tilde{P}$ a regular covering of P. The group π of deck-transformations of this covering acts on the vector-spaces $H_k(P; \mathbf{R})$. Let s_k be the character of the thus obtained representation of π on $H_k(P; \mathbf{R})$. Then:

$$b_k(P) = \frac{1}{|\pi|} \sum_{x \in \pi} s_k(x) \quad , \quad k = 0, 1, \dots$$

Here $|\pi|$ is the order of π. Now $s_k(x) \leqslant b_k(\tilde{P})$ since $b_k(\tilde{P})$ is the dimension of the representation. This gives the Bochner inequalities.

Proposition 3.1. Let M be an even-dimensional (*)-manifold. Then G is a centerless, compact and connected Lie group, and the identity component H_0 of an isotropy group is a maximal subgroup of maximal rank. (i.e. H_0 is not properly contained in a proper closed and connected subgroup of G.)

Proof: This follows from Proposition 1.2 and Lemma 1.2 together with the fact G/H_0 has positive characteristic. Q.E.D.

According to BOREL–DE SIEBENTHAL [2], the pairs (L, K) with L a compact, connected centerless simple Lie group, and K a closed, maximal connected subgroup of maximal rank, give rise to globally symmetric Riemannian manifolds, except in the following six cases: [1])

1. $(\mathbf{F}_4,\ \mathbf{A}_2 \times \mathbf{A}_2)$
2. $(\mathbf{E}_6,\ \mathbf{A}_2 \times \mathbf{A}_2 \times \mathbf{A}_2)$
3. $(\mathbf{E}_7,\ \mathbf{A}_2 \times \mathbf{A}_5)$
4. $(\mathbf{E}_8,\ \mathbf{A}_8)$
5. $(\mathbf{E}_8,\ \mathbf{A}_2 \times \mathbf{E}_6)$
6. $(\mathbf{E}_8,\ \mathbf{A}_4 \times \mathbf{A}_4)$

Lemma 3.2. The homogeneous spaces corresponding to the six exceptional cases listed above are not (*)-manifolds.

Proof: It is based on Bott's theorem. We calculate the real cohomology, i.e. the Poincaré-polynomials of the spaces in question with the aid of the Hirsch-formula. Denoting by $P(X, t)$ the Poincaré-polynomial over $\mathbf{R}$ of a space X, we have:

$P(\mathbf{F}_4, t) = (1+t^3)(1+t^{11})(1+t^{15})(1+t^{23})$

$P(\mathbf{E}_6, t) = (1+t^3)(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23})$

$P(\mathbf{E}_7, t) = (1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35})$

$P(\mathbf{E}_8, t) = (1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59})$

$P(\mathbf{A}_s, t) = (1+t^3)(1+t^5)(1+t^7)(1+t^9) \dots (1+t^{2s+1})$

[1]) The non-symmetric pair (G_2, A_2) gives the homogeneous space $G_2/A_2 = S_6$, which however has an essentially unique metric in which it is a twopoint homogeneous space.

The Poincaré-polynomials of the six spaces are found to be:

1. $(1+t^4)(1+t^6)^2(1+t^8)(1+t^{12})$
2. $(1+t^4)(1+t^6)(1+t^8)(1+t^{12})(1+t^2+t^4+t^6+t^8)(1+t^6+t^{12})(1-t^2+t^4)$
3. $(1+t^6)(1+t^8)(1+t^{10})(1+t^{12})(1+t^{18})(1+t^6+t^{12})(1+t^4+t^8+\ldots+t^{24})$
4. $(1+t^{12})(1+t^{14})(1+t^{18})(1+t^8+t^{16}+\ldots+t^{32})(1+t^6+t^{12}+\ldots+t^{42})$
   $(1+t^{10}+\ldots+t^{50})$
5. $(1+t^4+t^8+\ldots+t^{24})(1+t^6+\ldots+t^{54})(1+t^{12}+t^{24})(1+t^{10}+t^{20}+t^{30})$
   $(1+t^{18}+t^{36})$
6. $(1+t^8)(1+t^{16}+t^{32})(1+t^4+\ldots+t^{24})(1+t^6+\ldots+t^{30})(1+t^6+\ldots+t^{42})$
   $(1+t^{10}+t^{20}+t^{30})(1+t^{10}+\ldots+t^{50})$

From Bott's theorem we see that the Poincaré-polynomial over **R** of a (*)-manifold is of the form:

$$1+t^d+t^{2d}+\ldots+t^{rd} \qquad (d \text{ even})$$

for simply connected M; in the other case $b_k(M)=0$ for $k=1, \ldots, m-1$. This proves the lemma. Q.E.D.

Theorem 3.1. An even-dimensional two-point homogeneous manifold is a globally symmetric Riemannian manifold.

Proof: Lemma 3.2 and the preceding paragraph imply that G/H_0 is globally symmetric. Hence M is globally symmetric whenever H is connected, in particular when M is simply connected. The only case which is not settled is the case where H has two components. Then $G/H_0=\tilde{M}$, $\tilde{M}$ is globally symmetric, is a homology-sphere, and also is a (*)-manifold. But then $\tilde{M}$ must have rank 1, and so is isometric to a sphere. M itself is then necessarily a real projective space, and thus is globally symmetric. Q.E.D.

Katholieke Universiteit
Nijmegen, Netherlands and
Harvard University
Cambridge (Mass.), U.S.A.

REFERENCES

1. Borel, A., Some remarks about Lie groups transitive on spheres and tori, Bull. Amer. Math. Soc. **55**, 580–587 (1949).
2. ——— and J. de Siebenthal, Les sous-groupes fermés de rang maximum des groupes de Lie clos, Comment. Math. Helv. **23**, 202–221 (1949).
3. Bott, R., On Manifolds all of whose geodesics are closed, Ann. of Math. (2) **60**, 375–382 (1954).
4. Eckmann, B., Coverings and Betti-numbers, Bull. Amer. Math. Soc. **55**, 95–101 (1949).
5. Helgason, S., Differential Geometry and Symmetric spaces, Academic Press, New York, 1962.

6. HOPF, H. and H. SAMELSON, Ein Satz über die Wirkungsräume geschlossener Lieschen Gruppen, Comment. Math. Helv. **13**, 240–251 (1941).
7. OLIVIER, R., Die Existenz geschlossener Geodätischen auf kompakten Mannigfaltigkeiten, Comment. Math. Helv. **35**, 146–152 (1961).
8. WANG, H. C., Homogeneous spaces with non-vanishing Euler characteristic, Ann. of Math. (2) **50**, 925–953 (1949).
9. ———, A new characterization of spheres of even dimension, Nederl. Akad. Wetensch. **52**, 838–847 (1949).
10. ———, Two-point homogeneous spaces, Ann. of Math. (2) **55**, 177–191 (1952).
11. WHITEHEAD, J. H. C., Convex regions in the geometry of paths, Quart. J. Math. **3**, 33–42 (1932).

The following papers treat problems related to the one discussed here:

H. FREUDENTHAL, Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems, Math. Zeitschr. **63**, 374–405, (1956).

T. NAGANO, Homogeneous sphere-bundles and the isotropic Riemann manifolds, Nagoya Math. J. **15**, 29–55 (1959).